Musical Mathematics

ON THE ART AND SCIENCE OF ACOUSTIC INSTRUMENTS

Cris Forster

MUSICAL MATHEMATICS

ON THE ART AND SCIENCE OF ACOUSTIC INSTRUMENTS

MUSICAL MATHEMATICS

ON THE ART AND SCIENCE OF ACOUSTIC INSTRUMENTS

Text and Illustrations

by Cris Forster

CHRONICLE BOOKS

San Francisco

Library of Congress Cataloging-in-Publication Data available.

ISBN: 978-0-8118-7407-6

Manufactured in the United States.

All royalties from the sale of this book go directly to the Chrysalis Foundation, a public 501(c)3 nonprofit arts and education foundation.

www.chrysalis-foundation.org

Photo Credits:
Will Gullette, Plates 1–12, 14–16.
Norman Seeff, Plate 13.

10 9 8 7 6 5 4 3 2 1

Chronicle Books LLC
680 Second Street
San Francisco, California 94107

www.chroniclebooks.com

In Memory of Page Smith

my enduring teacher

And to Douglas Monsour

our constant friend

I would like to thank the following individuals and foundations for their generous contributions in support of the writing, designing, and typesetting of this work:

Peter Boyer and Terry Gamble-Boyer
The family of Jackson Vanfleet Brown
Thomas Driscoll and Nancy Quinn
Marie-Louise Forster
David Holloway
Jack Jensen and Cathleen O'Brien
James and Deborah Knapp
Ariano Lembi, Aidan and Yuko Fruth-Lembi
Douglas and Jeanne Monsour
Tim O'Shea and Peggy Arent
Fay and Edith Strange
Charles and Helene Wright

Ayrshire Foundation
Chrysalis Foundation

The jewel that we find, we stoop and take't,
Because we see it; but what we do not see
We tread upon, and never think of it.

W. Shakespeare

For more information about
Musical Mathematics: On the Art and Science of Acoustic Instruments
please visit:

www.chrysalis-foundation.org

www.amazon.com

www.chroniclebooks.com

CONTENTS

Foreword

I met Cris Forster more than thirty years ago. Shortly thereafter, I saw him perform *Song of Myself*, his setting of Walt Whitman poems from *Leaves of Grass*. His delivery was moving and effective. Several of the poems were accompanied by his playing on unique instruments — one an elegant box with many steel strings and moveable bridges, a bit like a koto in concept; the other had a big wheel with strings like spokes from offset hubs, and he rotated the wheel as he played and intoned the poetry. I was fascinated.

Since that time, Cris has built several more instruments of his own design. Each shows exquisite care in conception and impeccable craftsmanship in execution. And of course, they are a delight to hear. Part of what makes them sound so good is his deep understanding of how acoustic musical instruments work, and part is due to his skill in working the materials to his exacting standards.

But another important aspect of their sound, and indeed one of the main reasons Cris could not settle for standard instruments, is that his music uses scales and harmonies that are not found in the standard Western system of intonation (with each octave divided into twelve equal semitones, called equal temperament). Rather, his music employs older notions of consonance, which reach back as far as ancient Greek music and to other cultures across the globe, based on what is called just intonation. Here, the musical intervals that make up the scales and chords are those that occur naturally in the harmonic series of overtones, in stretched flexible strings, and in organ pipes, for example.

In just intonation, the octave is necessarily divided into unequal parts. In comparison to equal temperament, the harmonies of just intonation have been described as smoother, sweeter, and/or more powerful. Many theorists consider just intonation to be the standard of comparison for consonant intervals. There has been a resurgence of interest in just intonation since the latter part of the twentieth century, spurred by such pioneers as Harry Partch and Lou Harrison. Even so, the community of just intonation composers remains comparatively quite small, and the subset of those who employ only acoustic instruments is much smaller still. I know of no other living composer who has created such a large and varied ensemble of high-quality just intoned acoustical instruments, and a body of music for them, as Cris Forster.

Doing what he has done is not easy, far from it. The long process of developing his instruments has required endless experimentation and careful measurement, as well as intense study of the literature on acoustics of musical instruments. In this way Cris has developed deep and rich knowledge of how to design and build instruments that really work. Also, in the service of his composing, Cris has studied the history of intonation practices, not only in the Western tradition, but around the world.

This book is his generous offering of all that hard-earned knowledge, presented as clearly as he can make it, for all of you who have an interest in acoustic musical instrument design and/or musical scales over time and space. The unifying theme is how mathematics applies to music, in both the acoustics of resonant instruments and the analysis of musical scales. The emphasis throughout is to show how to use these mathematical tools, without requiring any background in higher mathematics; all that is required is the ability to do arithmetic on a pocket calculator, and to follow Cris' clear step-by-step instructions and examples. Any more advanced mathematical tools required, such as logarithms, are carefully explained with many illustrative examples.

The first part of the book contains practical information on how to design and build musical instruments, starting from first principles of vibrating sound sources of various kinds. The ideas are explained clearly and thoroughly. Many beautiful figures have been carefully conceived to illuminate the concepts. And when Cris gives, say, formulas for designing flutes, it's not just something he read in a book somewhere (though he has carefully studied many books); rather, you can be sure it is something he has tried out: he knows it works from direct experience. While some of this information can be found (albeit in a less accessible form) in other books on musical acoustics, other information appears nowhere else. For example, Cris developed a method for tuning the overtones of marimba bars that results in a powerful, unique tone not found in commercial instruments. Step-by-step instructions are given for applying this technique (see Chapter 6). Another innovation is Cris' introduction of a new unit of mass, the "mica," that greatly simplifies calculations using lengths measured in inches. And throughout Cris gives careful explanations, in terms of physical principles, that make sense based on one's physical intuition and experience.

The latter part of the book surveys the development of musical notions of consonance and scale construction. Chapter 10 traces Western ideas about intonation, from Pythagoras finding number in harmony, through "meantone" and then "well-temperament" in the time of J.S. Bach, up to modern equal temperament. The changing notions of which intervals were considered consonant when, and by whom, make a fascinating story. Chapter 11 looks at the largely independent (though sometimes parallel) development of musical scales and tunings in various Eastern cultures, including China, India, and Indonesia, as well as Persian, Arabian, and Turkish musical traditions. As far as possible, Cris relies on original sources, to which he brings his own analysis and explication. To find all of these varied scales compared and contrasted in a single work is unique in my experience.

The book concludes with two short chapters on specific original instruments. One introduces the innovative instruments Cris has designed and built for his music. Included are many details of construction and materials, and also scores of his work that demonstrate his notation for the instruments. The last chapter encourages the reader (with explicit plans) to build a simple stringed instrument (a "canon") with completely adjustable tuning, to directly explore the tunings discussed in the book. In this way, the reader can follow in the tradition of Ptolemy, of learning about music through direct experimentation, as has Cris Forster.

David R. Canright, Ph.D.
Del Rey Oaks, California
January 2010

Introduction and Acknowledgments

In simplest terms, human beings identify musical instruments by two aural characteristics: a particular kind of sound or timbre, and a particular kind of scale or tuning. To most listeners, these two aspects of musical sound do not vary. However, unlike the constants of nature — such as gravitational acceleration on earth, or the speed of sound in air — which we cannot change, the constants of music — such as string, percussion, and wind instruments — are subject to change. A creative investigation into musical sound inevitably leads to the subject of musical mathematics, and to a reexamination of the meaning of variables.

The first chapter entitled "Mica Mass" addresses an exceptionally thorny subject: the derivation of a unit of mass based on an inch constant for acceleration. This unit is intended for builders who measure wood, metal, and synthetic materials in inches. For example, with the mica unit, builders of string instruments can calculate tension in pounds-force, or lbf, without first converting the diameter of a string from inches to feet. Similarly, builders of tuned bar percussion instruments who know the modulus of elasticity of a given material in pounds-force per square inch, or lbf/in^2, need only the mass density in mica/in^3 to calculate the speed of sound in the material in inches per second; a simple substitution of this value into another equation gives the mode frequencies of uncut bars.

Chapters 2–4 explore many physical, mathematical, and musical aspects of strings. In Chapter 3, I distinguish between four different types of ratios: ancient length ratios, modern length ratios, frequency ratios, and interval ratios. Knowledge of these ratios is essential to Chapters 10 and 11. Many writers are unaware of the crucial distinction between ancient length ratios and frequency ratios. Consequently, when they attempt to define arithmetic and harmonic divisions of musical intervals based on frequency ratios, the results are diametrically opposed to those based on ancient length ratios. Such confusion leads to anachronisms, and renders the works of theorists like Ptolemy, Al-Fārābī, Ibn Sīnā, and Zarlino incomprehensible.

Chapter 5 investigates the mechanical interactions between piano strings and soundboards, and explains why the large physical dimensions of modern pianos are not conducive to explorations of alternate tuning systems.

Chapters 6 and 7 discuss the theory and practice of tuning marimba bars and resonators. The latter chapter is essential to Chapter 8, which examines a sequence of equations for the placement of tone holes on concert flutes and simple flutes.

Chapter 9 covers logarithms, and the modern cent unit. This chapter serves as an introduction to calculating scales and tunings discussed in Chapters 10 and 11.

In summary, this book is divided into three parts. (1) In Chapters 1–9, I primarily examine various vibrating systems found in musical instruments; I also focus on how builders can customize their work by understanding the functions of variables in mathematical equations. (2) In Chapter 10, I discuss scale theories and tuning practices in ancient Greece, and during the Renaissance and Enlightenment in Europe. Some modern interpretations of these theories are explained as well. In Chapter 11, I describe scale theories and tuning practices in Chinese, Indonesian, and Indian music, and in Arabian, Persian, and Turkish music. For Chapters 10 and 11, I consistently studied original texts in modern translations. I also translated passages in treatises by Ptolemy, Al-Kindī, the Ikhwān al-Ṣafā, Ibn Sīnā, Stifel, and Zarlino from German into English; and in collaboration with two contributors, I participated in translating portions of works by Al-Fārābī, Ibn Sīnā, Ṣafī Al-Dīn, and Al-Jurjānī from French into English. These translations reveal that all the above-mentioned theorists employ the language of ancient length ratios. (3) Finally, Chapters 12 and 13 recount musical instruments I have built and rebuilt since 1975.

I would like to acknowledge the assistance and encouragement I received from Dr. David R. Canright, associate professor of mathematics at the Naval Postgraduate School in Monterey, California. David's unique understanding of mathematics, physics, and music provided the foundation for many conversations throughout the ten years I spent writing this book. His mastery of differential equations enabled me to better understand dispersion in strings, and simple harmonic motion of air particles in resonators. In Section 4.5, David's equation for the effective length of stiff strings is central to the study of inharmonicity; and in Section 6.6, David's figure, which shows the effects of two restoring forces on the geometry of bar elements, sheds new light on the physics of vibrating bars. Furthermore, David's plots of compression and rarefaction pulses inspired numerous figures in Chapter 7. Finally, we also had extensive discussions on Newton's laws. I am very grateful to David for his patience and contributions.

Heartfelt thanks go to my wife, Heidi Forster. Heidi studied, corrected, and edited myriad versions of the manuscript. Also, in partnership with the highly competent assistance of professional translator Cheryl M. Buskirk, Heidi did most of the work translating extensive passages from *La Musique Arabe* into English. To achieve this accomplishment, she mastered the often intricate verbal language of ratios. Heidi also assisted me in transcribing the Indonesian and Persian musical scores in Chapter 11, and transposed the traditional piano score of "The Letter" in Chapter 12. Furthermore, she rendered invaluable services during all phases of book production by acting as my liaison with the editorial staff at Chronicle Books. Finally, when the writing became formidable, she became my sparring partner and helped me through the difficult process of restoring my focus. I am very thankful to Heidi for all her love, friendship, and support.

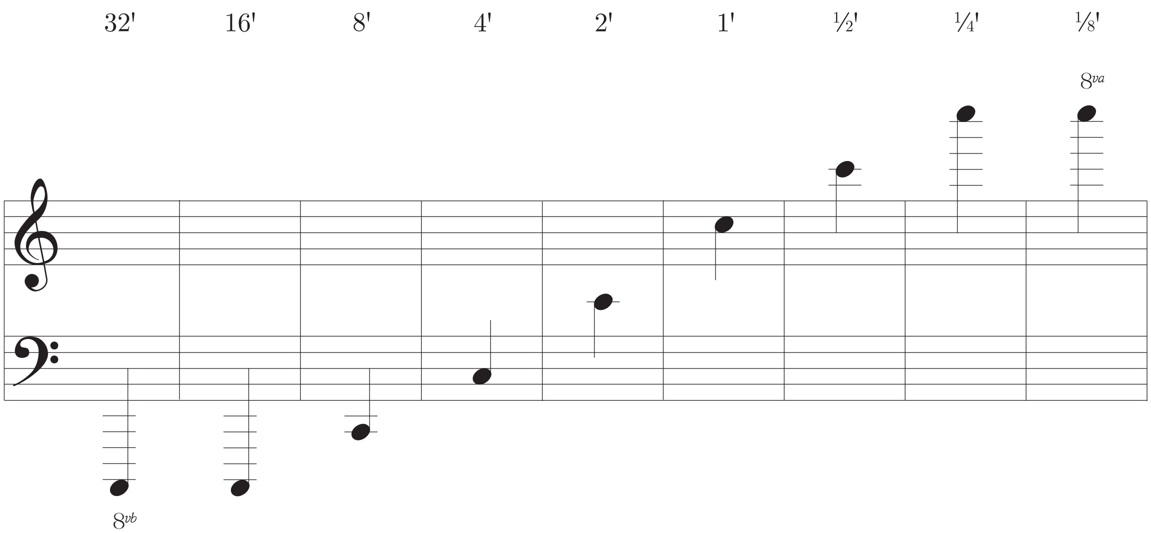
I would also like to express my appreciation to Dr. John H. Chalmers. Since 1976, John has generously shared his vast knowledge of scale theory with me. His mathematical methods and techniques have enabled me to better understand many historical texts, especially those of the ancient Greeks. And John's scholarly book *Divisions of the Tetrachord* has furthered my appreciation for world tunings.

I am very grateful to Lawrence Saunders, M.A. in ethnomusicology, for reading Chapters 3, 9, 10, and 11, and for suggesting several technical improvements.

Finally, I would like to thank Will Gullette for his twelve masterful color plates of the Original Instruments and String Winder, plus three additional plates. Will's skill and tenacity have illuminated this book in ways that words cannot convey.

Cris Forster
San Francisco, California
January 2010

TONE NOTATION



1.	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
2.	$C_{,,}$	$C_{,}$	C	c	c'	c''	c'''	c''''	c^{V}
3.	C_2	C_1	C_0	c^0	c^1	c^2	c^3	c^4	c^5

1. American System, used throughout this text.
2. Helmholtz System.
3. German System.

LIST OF SYMBOLS

Latin

12-TET	12-tone equal temperament
a	Acceleration; in/s^2
a.l.r.	Ancient length ratio; dimensionless
B	Bending stiffness of bar; lbf·in^2, or mica·in^3/s^2
B'	Bending stiffness of plate; lbf·in, or mica·in^2/s^2
B_{A}	Adiabatic bulk modulus; psi, lbf/in^2, or mica/(in·s^2)
B_{I}	Isothermal bulk modulus; psi, lbf/in^2, or mica/(in·s^2)
b	Width; in
¢	Cent, 1/100 of a "semitone," or 1/1200 of an "octave"; dimensionless
$\bar{¢}$	Coefficient of inharmonicity of string; cent
c_{B}	Bending wave speed; in/s
c_{L}	Longitudinal wave speed, or speed of sound; in/s
c_{T}	Transverse wave speed; in/s
c.d.	Common difference of an arithmetic progression; dimensionless
c.r.	Common ratio of a geometric progression; dimensionless
cps	Cycle per second; 1/s
D	Outside diameter; in
D_{i}	Inside diameter of wound string; in
D_{m}	Middle diameter of wound string; in
D_{o}	Outside diameter of wound string; in
D_{w}	Wrap wire diameter of wound string; in
d	Inside diameter, or distance; in
E	Young's modulus of elasticity; psi, lbf/in^2, or mica/(in·s^2)
F	Frequency; cps
F_{c}	Critical frequency; cps
F_{n}	Resonant frequency; cps
$\bar{F}_{\mathrm{n}}$	Inharmonic mode frequency of string; cps
f	Force; lbf, or mica·in/s^2
f.r.	Frequency ratio; dimensionless
g	Gravitational acceleration; 386.0886 in/s^2
h	Height, or thickness; in
I	Area moment of inertia; in^4
i.r.	Interval ratio; dimensionless
J	Stiffness parameter of string; dimensionless
K	Radius of gyration; in
k	Spring constant; lbf/in, or mica/s^2
L	Length; in, cm, or mm
ℓ_{M}	Multiple loop length of string; in
ℓ_{S}	Single loop length of string; in
l.r.	Length ratio; dimensionless
lbf	Pounds-force; mica·in/s^2
lbm	Pounds-mass; 0.00259008 mica

$M/u.a.$	Mass per unit area; mica/in^2, or lbf·s^2/in^3
$M/u.l.$	Mass per unit length; mica/in, or lbf·s^2/in^2
m	Mass; mica, or lbf·s^2/in
n	Mode number, or harmonic number; any positive integer
P	Pressure; psi, lbf/in^2, or mica/(in·s^2)
p	Excess acoustic pressure; psi, lbf/in^2, or mica/(in·s^2)
psi	Pounds-force per square inch; lbf/in^2, or mica/(in·s^2)
q	Bar parameter; dimensionless
R	Ideal gas constant; in·lbf/(mica·°R), or in^2/(s^2·°R)
r	Radius; in
S	Surface area; in^2
SHM	Simple harmonic motion
T	Tension; lbf, or mica·in/s^2
T_{A}	Absolute temperature; dimensionless
t	Time; s
U	Volume velocity; in^3/s
u	Particle velocity; in/s
V	Volume; in^3
v	Phase velocity; in/s
W	Weight density, or weight per unit volume; lbf/in^3, or mica/(in^2·s^2)
w	Weight; lbf, or mica·in/s^2
Y_{A}	Acoustic admittance; in^4·s/mica
Z_{A}	Acoustic impedance; mica/(in^4·s)
Z_{r}	Acoustic impedance of room; mica/(in^4·s)
Z_{t}	Acoustic impedance of tube; mica/(in^4·s)
Z_{M}	Mechanical impedance; mica/s
Z_{b}	Mechanical impedance of soundboard; mica/s
Z_{p}	Mechanical impedance of plate; mica/s
Z_{s}	Mechanical impedance of string; mica/s
Z_{R}	Radiation impedance; mica/s
Z_{a}	Radiation impedance of air; mica/s
z	Specific acoustic impedance; mica/(in^2·s)
z_{a}	Characteristic impedance of air; 0.00153 mica/(in^2·s)

Greek

$\mathit{\Delta}$	Correction coefficient, or end correction coefficient; dimensionless
$\Delta\ell$	Correction, or end correction; in, cm, or mm
δ	Departure of tempered ratio from just ratio; cent
γ	Ratio of specific heat; dimensionless
θ	Angle; degree
κ	Conductivity; in
$\mathit{\Lambda}$	Bridged canon string length; in
$\mathit{\Lambda}_{\mathrm{A}}$	Arithmetic mean string length; in
$\mathit{\Lambda}_{\mathrm{G}}$	Geometric mean string length; in
$\mathit{\Lambda}_{\mathrm{H}}$	Harmonic mean string length; in

λ	Wavelength; in
λ_B	Bending wavelength; in
λ_L	Longitudinal wavelength; in
λ_T	Transverse wavelength; in
μ	Poisson's ratio; dimensionless
Π	Fretted guitar string length; mm
π	Pi; ≈ 3.1416
ρ	Mass density, or mass per unit volume; mica/in^3, or lbf·s^2/in^4
τ	Period, or second per cycle; s

Index

Internet Version: no diacritics

B

Q

R

T

U

Y

Z